PROJECTS – TOPICS IN MODERN GEOMETRY

Algebra of tangent space

In a similar manner to the correspondence between varieties and coordinate rings, we can define a correspondence between the tangent space at a point and a certain vector space. *Basic algebraic geometry, Shafarevich: (pages 86–88).*

Bézout's theorem

In this project, you will give a summary of Bézout's theorem, and you will prove a weak form of Bézout's theorem, which says that for any two complex planar curves of degree m and n, the number of intersections is at most mn.

You will read §3.1 of *Complex Algebraic Curves* by Kirwan (pages 51–69), paying particular attention to Theorem 3.9 and its proof on pages 54–55. You will adapt the proof to hold in $\mathbb{C}^2$ (the proof in the book is in the projective plane $\mathbb{P}^2(\mathbb{C})$), and summarise the ideas of the proof of the full version of Bézout's theorem.

Birational equivalence

In this project, you will investigate the notion of *birational equivalence*, which is a weaker condition than isomorphism: for example, the desingularisation of a singular curve is birationally equivalent to the original curve, but not isomorphic to it.

You will read §7.2 and §7.3 of *An Invitation to Algebraic Geometry* by Smith et al (pages 112–115), and provide solutions to some of the exercises.

Blowing up subvarieties

In this project, you will generalise from the notion of blowing up a point on a variety to the idea of blowing up a *subvariety* of a variety (a point on a variety being the simplest example of a subvariety).

You will read §7.4 of *An Invitation to Algebraic Geometry* by Smith et al (pages 115–119), and summarise the process of blowing up a subvariety of a variety. You will then apply this technique to desingularise a specific singular surface by blowing up a singular subvariety.

Grassmannians

In this project, you will investigate *Grassmannians*, which are spaces generalising the notion of a projective space.

You will read §5.4 of *An Invitation to Algebraic Geometry* by Smith et al (pages 71–74), write a summary of the theory of Grassmannians presented there, and provide a solution to Exercise 5.4.1.

Gröbner bases

A Gröbner basis for an ideal is a particular basis that allows for computational efficiency. You may wish to define a Gröbner basis and demonstrate how it can be used with a division algorithm and/or the ideal membership problem. *Ideals, Varieties and Algorithms, Cox, Little, O'Shea (Chapter 2, sections 1–3).*

The group law on non-singular cubics

In this project, you will look at how we can define an operation on the rational points on a non-singular cubic curve to form a group.

You will read §7 of *Lectures on elliptic curves* by Cassels (pages 27–31), summarise the material there, and provide solutions to some of the exercises.

Localisation

One technique to understanding the local structure of a variety is via localisation of coordinate rings. Review *Undergraduate commutative algebra, Reid (Chapter 6, sections 6.1–6.4)*.

Prime factorisation using elliptic curves

In this project, you will investigate how elliptic curves can be powerful tools for factorising large integers.

You will read §26 of *Lectures on elliptic curves* by Cassels (pages 124–129), summarise the material there, and provide solutions to some of the exercises.

The projective closure of a variety

In this project, you will develop the notion of *projective varieties* and the *projective closure* of an affine variety.

You will read §3.1 and §3.2 of *An Invitation to Algebraic Geometry* by Smith et al (pages 37–44), and summarise the basic theory of projective varieties and projective closures. You will then use this to take the projective closure of two specific affine curves, and calculate the number of intersections between them.

Projective plane geometry

In this project, you will go into more detail about the *geometry* of the projective plane.

You will read Nigel Hitchin's online notes on projective geometry (22 pages), paying particular attention to §2.2 (on linear subspaces) and §2.4 (on duality). You will write a summary of the geometry of the projective plane, giving proofs of certain key results.

Proving Zariski's lemma

This will complete the proof of Hilbert's Nullstellensatz involving mathematics of finite algebras and Noether normalisation. *Undergraduate algebraic geometry, Reid (3.12–3.16, pages 57–62).*

Spectrum of a ring

We can impose a geometry on an arbitrary ring via its spectrum. This uses ideas from the weak Nullstellensatz that gives a correspondence between points and maximal ideals. *An invitation to algebraic geometry, Kahanpää, Kekäläinen, Smith and Traves (section 2.6, pages 30–32).*

Sheaves

An ambitious project not for the faint hearted. A sheaf is a tool for understanding local information attached to a topological space, such as a variety.

27 lines on a cubic surface

In this project, you will investigate the very well-known result that any nonsingular cubic surface in $\mathbb{P}^3(\mathbb{C})$ contains exactly 27 lines.

You will read §7 of *Undergraduate algebraic geometry* by Reid (pages 102–113), summarise the materia there, and provide solutions to some of the exercises.

The Zariski topology

You will define a topological space and show that the Zariski topology satisfies this definition. Provide some examples on the structure of this topology: for example you may wish to answer exercises 1.2.2, 2.3.4 and 2.3.5 of *An invitation to algebraic geometry, Kahanpää, Kekäläinen, Smith and Traves*, or consider if any of the separation axioms hold.