LECTURE 8: PROJECTIVE SPACES AND PROJECTIVE GEOMETRY

Projective geometry concerns itself with extending a vector space by in some sense adding an infinitely distant point in every direction.

1. The projective line

Definition 1. *Let k be a field. Then the projective line $\mathbb{P}^1(k)$ is the set of all lines through the origin in the vector space k^2.*

How can we think about $\mathbb{P}^1(k)$?

1.1. Intersections with a line.

We let $k = \mathbb{R}$, and consider the line $x = 1$ in $\mathbb{R}^2$. For any $m \in \mathbb{R}$, the line $y = mx$ through the origin will intersect $y = 1$ at exactly one point (the point $(1, m)$), and hence the projective line $\mathbb{P}^1(\mathbb{R})$ contains a copy of the standard real line $\mathbb{R}$. There is only one more line through the origin which we have not accounted for: namely, the line $x = 0$. This does *not* intersect the line $x = 1$, and hence we have an extra point in $\mathbb{P}^1(\mathbb{R})$, which we call the *point at infinity*, and denote by ∞.

Remark. This construction will work for any field k; it's just easier to picture with $k = \mathbb{R}$.

Note that as the slope m of a line becomes large and positive, the point it corresponds to on the line $x = 1$ is further and further up the line, and that as the slope becomes large and negative, the corresponding point is further and further *down* the line. However, lines with large positive and large negative slope both limit to the same line: namely, the vertical line $x = 0$. As a result, the corresponding point at infinity on $\mathbb{P}^1(\mathbb{R})$ can be simultaneously thought of as infinitely far up *and* infinitely far down the line $x = 1$, and so the projective line $\mathbb{P}^1(\mathbb{R})$ can be thought of as behaving like a circle. This can be more clearly seen in the next visualisation.

1.2. Intersections with a half-circle.

Again, let $k \in \mathbb{R}$, and now consider the circle

$$S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.$$

Each line through the origin will intersect this circle exactly twice, at two opposite points of the circle. So if we restrict to the half-circle

$$\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1, y \geq 0,$$

then all but one line intersects this at a unique point: the only exception is the horizontal line $y = 0$, which intersects at $(-1, 0)$ and $(1, 0)$. Similarly to before, lines with very small positive slope intersect the circle very close to $(1, 0)$, and lines with very small negative slope intersect the circle very close to $(-1, 0)$, again giving us the idea that the two points $(1, 0)$ and $(-1, 0)$ should be identified and that $\mathbb{P}^1(\mathbb{R})$ should be thought of as a circle.

1.3. **Quotient space.** The projective line $\mathbb{P}^1(k)$ can also be described as a *quotient space* of $k^2\backslash\{\mathbf{0}\}$. We want two points of k^2 to be equal if they lie on the same line through the origin; so if we exclude the origin $\mathbf{0}$ (which lies on each of these lines by definition), we have an equivalence relation $\sim$ defined by

$$(x_0, x_1) \sim (y_0, y_1) \;\Leftrightarrow\; (y_0, y_1) = (\lambda x_0, \lambda x_1) \text{ for some } \lambda \in k^\times.$$

Exercise. Verify that this is an equivalence relation on $k^2\backslash\{\mathbf{0}\}$.

Then we can define $\mathbb{P}^1(k)$ to be the quotient space

$$\mathbb{R}^2\backslash\{\mathbf{0}\}/\sim.$$

We tend to write elements of $\mathbb{P}^1(k)$ as $[x_0 : x_1]$, and we can always scale by non-zero elements of k.

Example. Let $k = \mathbb{C}$. Then we have

$$[i : 3+i] = [-1 : 3i-1] = [2i : 6+2i] = \text{many other examples}$$

Note that if an element $[x_0 : x_1] \in \mathbb{P}^1(k)$ has $x_0 \neq 0$, then we can write

$$[x_0 : x_1] = [1 : \tfrac{x_1}{x_0}] = [1 : z_0],$$

and if $x_0 = 0$, we have

$$[0 : x_1] = [0 : 1]$$

(if $x_0 = 0$ then $x_1 \neq 0$, since we cannot have $(x_0, x_1) = (0, 0)$, so we can divide through by x_1). So we have a decomposition of $\mathbb{P}^1(k)$ as

$$\mathbb{P}^1(k) = k \cup \{\infty\}.$$

We could also have gotten this decomposition by considering the cases $x_1 \neq 0$ and $x_1 = 0$. We call the sets

$$U_0 = \{[x_0 : x_1] \in \mathbb{P}^1(k) \mid x_0 \neq 0\} \cong k$$

and

$$U_1 = \{[x_0 : x_1] \in \mathbb{P}^1(k) \mid x_1 \neq 0\} \cong k$$

the *affine charts* of $\mathbb{P}^1(k)$.

2. Projective spaces

Definition 2. *Let k be a field. Then projective n-space $\mathbb{P}^n(k)$ is the set of all lines through the origin in the vector space k^{n+1}.*

As before, this can be visualised in various ways: via the intersections of the lines with a fixed hyperplane in k^{n+1}, or via the intersections of the lines with a half-sphere in k^{n+1}.

We describe $\mathbb{P}^n(k)$ as a quotient space of $k^{n+1}\backslash\{\mathbf{0}\}$ by the equivalence relation

$$(x_0, \ldots, x_n) \sim (y_0, \ldots, y_n) \;\Leftrightarrow\; (y_0, \ldots, y_n) = (\lambda x_0, \ldots, \lambda x_n) \text{ for some } \lambda \in k^\times.$$

Exercise. As before, verify that this is an equivalence relation on $k^{n+1}\backslash\{\mathbf{0}\}$.

We write elements of $\mathbb{P}^n(k)$ as

$$[x_0 : x_1 : \cdots : x_n].$$

As before, for each $i = 0, 1, \ldots, n$, we have an *affine chart* of $\mathbb{P}^n(k)$ given by

$$U_i = \{[x_0 : x_1 : \cdots : x_n] \in \mathbb{P}^n(k) \mid x_i \neq 0\} \cong k^n.$$

Now, consider separately the cases where $x_0 \neq 0$ and $x_0 = 0$. If $x_0 \neq 0$, then we are on the affine chart U_0, which is isomorphic to k^n (since now all of the remaining

x_i can be zero; if $x_0 = 0$, then at least one of the remaining $x_1, \ldots, x_n$ must be non-zero, so we have something isomorphic to $\mathbb{P}^{n-1}(k)$.

So we have

$$\mathbb{P}^n(k) = k^n \cup \mathbb{P}^{n-1}(k),$$

and hence (by induction) we have

$$\mathbb{P}^n(k) = k^n \cup k^{n-1} \cup \cdots \cup k \cup \{\infty\}.$$

3. Projective varieties and the projective closure of a variety

We will not develop the theory of projective varieties or projective closures in any real detail: we will just give the broad brushstrokes, to give an idea of what can be done in these spaces. (Filling out these details could form the basis of one or more projects.)

First, we note that a polynomial will not take a consistent value on an equivalence class in a projective space: that is, polynomial functions will not be *well-defined*. For example: in $\mathbb{P}^1(\mathbb{R})$, we have

$$[1:1] = [2:2],$$

but if we look at (for example) the function $p(X, Y) = X^2 + Y^2 - 1$, we have

$$p(1,1) = 1 \neq 7 = p(2,2).$$

So p does not take consistent values on the equivalence class $[1:1]$.

However, consider *homogeneous polynomials*: those where all terms have the same degree.

Example. $XYZ - Z^3 + X^2Y$ is a homogeneous polynomial of degree 3, but $XYZ - X$ is not (it has one term of degree 3 and one term of degree 1).

The values of a homogeneous polynomial in projective space will still not be well-defined: however, its *zero set* will be. If $F \in k[X_0, \ldots, X_n]$ is a homogeneous polynomial of degree d, then we have

$$F(\lambda x_0, \ldots, \lambda x_n) = \lambda^d F(x_0, \ldots, x_n),$$

and hence we have

$$F(\lambda x_0, \ldots, \lambda x_n) = 0 \;\Leftrightarrow\; F(x_0, \ldots, x_n) = 0.$$

So we can define a *projective variety* in $\mathbb{P}^n(k)$ to be the common zero set of a collection of *homogeneous* polynomials in $k[x_0, \ldots, x_n]$.

There exists a homogeneous version of the Nullstellensatz, stating that the projective subvarieties of $\mathbb{P}^n(k)$ are in one-to-one correspondence with the radical ideals of $k[X_0, \ldots, X_n]$ which admit a set of homogeneous generators (except for the ideal $(X_0, \ldots, X_n)$, which generates the origin in k^{n+1}).

If we intersect a projective variety in $\mathbb{P}^n(k)$ with any of the $n+1$ affine charts of $\mathbb{P}^n(k)$, we get an affine variety in k^n. Conversely, if we have an affine variety in k^n, we can form a projective variety in $\mathbb{P}^n(k)$ which is in some sense a completion of our original variety: a projective variety whose intersection with an affine chart of $\mathbb{P}^n(k)$ gives the variety we started with.