LECTURE 2: POLYNOMIAL RINGS AND AFFINE VARIETIES

In this lecture, we will reintroduce *polynomial rings* over the complex numbers $\mathbb{C}$, and use these to define the concept of an *affine algebraic variety* over $\mathbb{C}$. We will then give some examples and non-examples of varieties, and show some basic properties of varieties.

1. Polynomial rings over $\mathbb{C}$

Recall. We have

$$\mathbb{C}[X] = \left\{ a_0 + a_1 X + a_2 X^2 + \cdots + a_m X^m \;\middle|\; a_i \in \mathbb{C},\ m \in \mathbb{Z}_{\geq 0} \right\}.$$

We have

$$\mathbb{C}[X,Y] = \left\{ \sum_{i=0}^{\ell} \sum_{j=0}^{m} a_{ij} X^i Y^j \;\middle|\; a_{ij} \in \mathbb{C},\ \ell, m \in \mathbb{Z}_{\geq 0} \right\},$$

and more generally,

$$\mathbb{C}[X_1, \ldots, X_n] = \left\{ \sum_{i_1=0}^{m_1} \cdots \sum_{i_n=0}^{m_n} a_{i_1 \cdots i_n} X_1^{i_1} \cdots X_n^{i_n} \;\middle|\; a_{i_1 \cdots i_n} \in \mathbb{C},\ m_1, \ldots, m_n \in \mathbb{Z}_{\geq 0} \right\}.$$

We will go into more detail about the properties of these rings in Lectures 4–6. Right now, all we need is the definition.

2. Affine varieties

Definition 1. *An* affine algebraic variety *is the set of common zeros in $\mathbb{C}^n$ of a set of polynomials in $\mathbb{C}[X_1, \ldots, X_n]$. For a set*

$$\{F_i \in \mathbb{C}[X_1, \ldots, X_n] \mid i \in I\}$$

of polynomials (where I is some indexing set), we have

$$V = \{(x_1, \ldots, x_n) \in \mathbb{C}^n \mid F_i(x_1, \ldots, x_n) = 0 \;\; \textit{for all} \;\; i \in I\}.$$

We usually write

$$V = \mathbb{V}(\{F_i\}_{i \in I}) \subset \mathbb{C}^n.$$

Important note! From now on, we will simply refer to affine algebraic varieties as *varieties*, for the sake of brevity. However, be aware that this is *not* the most general definition of variety, and hence when you see the word "variety" in other sources, they may be talking about something more general.

3. Examples and non-examples of varieties

Examples of varieties.

(1) The entire space $\mathbb{C}^n$ and the empty set $\emptyset$ are both varieties: we have

$$\mathbb{C}^n = \mathbb{V}(0) \quad \text{and} \quad \emptyset = \mathbb{V}(1).$$

(2) A set containing a single point $(a_1, \ldots, a_n) \in \mathbb{C}^n$ is a variety:
$$\{(a_1, \ldots, a_n)\} = \mathbb{V}(X_1 - a_1, \ldots, X_n - a_n).$$
(3) We can consider
$$\mathbb{V}(X^2 + Y^2 - 1) \subset \mathbb{C}^2,$$
whose real locus is the unit circle in $\mathbb{R}^2$.
(4) Consider
$$\mathbb{V}(y^2 - x^3 + x - 1).$$
This is an *elliptic curve*: we can draw its real locus.
(5) The set $\mathrm{SL}_2(\mathbb{C})$ can be identified with a variety in $\mathbb{C}^4$, given by
$$\mathbb{V}(X_1X_4 - X_2X_3 - 1) \subset \mathbb{C}^4.$$
Similarly, the set of 2×2 matrices of any fixed determinant $D \in \mathbb{C}$ form a variety in $\mathbb{C}^4$ given by
$$\mathbb{V}(X_1X_4 - X_2X_3 - D) \subset \mathbb{C}^4.$$

Non-examples.

(1) A closed square in $\mathbb{C}^2$, given by
$$\{(x, y) \in \mathbb{C}^2 \mid |x| \leq 1, |y| \leq 1\},$$
is not a variety. (We will go into why in the first problems class.)
(2) The graph of a transcendental function is not a variety. For example, the zero set of $y - e^x$ is not a variety.
(3) The set $\mathrm{GL}_2(\mathbb{C})$ cannot be viewed as a variety. To quickly see why, we will develop some basic properties of varieties.

4. Some properties of varieties

Proposition 2. (i) *The union of finitely many varieties in $\mathbb{C}^n$ is a variety in $\mathbb{C}^n$.*
(ii) *The intersection of arbitrarily many varieties in $\mathbb{C}^n$ is a variety in $\mathbb{C}^n$.*

Proof. (i) We show this result is true for the union of two varieties: then it follows for the union of finitely many varieties by induction.

Suppose we have
$$V = \mathbb{V}(F) \quad \text{and} \quad W = \mathbb{V}(G).$$
Then by definition we have
$$V = \{z \in \mathbb{C}^n \mid F(z) = 0\},$$
$$W = \{z \in \mathbb{C}^n \mid G(z) = 0\}.$$
So we want
$$V \cup W = \{z \in \mathbb{C}^n \mid F(z) = 0 \text{ or } G(z) = 0\}.$$
Then notice that the polynomial FG vanishes at a point p if and only if at least one of F and G vanishes at p, which is exactly what we want. So we have
$$V \cup W = \{z \in \mathbb{C}^n \mid F(z)G(z) = 0\} = \mathbb{V}(FG).$$
More generally, but by exactly the same principle, if we have
$$V = \mathbb{V}(\{F_i\}_{i\in I}) \quad \text{and} \quad W = \mathbb{V}(\{G_j\}_{j\in J}),$$

then we have

$$V \cup W = \mathbb{V}(\{F_i G_j\}_{(i,j)\in I\times J}).$$

(ii) Let

$$V = \mathbb{V}(\{F_i\}_{i\in I}) \quad \text{and} \quad W = \mathbb{V}(\{F_j\}_{j\in J}).$$

Then by definition we have

$$V = \{z \in \mathbb{C}^n \mid F_i(z) = 0 \text{ for all } i \in I\},$$
$$W = \{z \in \mathbb{C}^n \mid F_j(z) = 0 \text{ for all } j \in J\}.$$

But then

$$\begin{aligned} V \cap W &= \{z \in \mathbb{C}^n \mid F_i(z) = 0 \text{ for all } i \in I \text{ and } F_j(z) = 0 \text{ for all } j \in J\} \\ &= \{z \in \mathbb{C}^n \mid F_i(z) = 0 \text{ for all } i \in I \cup J\} \\ &= \mathbb{V}(\{F_i\}_{i\in I\cup J}). \end{aligned}$$

□

Project idea: Zariski topology and affine space. These properties can be used to define a topology on $\mathbb{C}^n$, called the *Zariski topology*, and an associated topological space called *affine n-space*, denoted by $\mathbb{A}^n$.

Proposition 3. *Every variety in $\mathbb{C}^n$ is closed in the Euclidean topology.*

Proof. Let

$$V = \mathbb{V}(\{F_i\}_{i\in I}).$$

We showed in the proof of the previous proposition that we have

$$V = \bigcap_{i\in I} \mathbb{V}(F_i).$$

The intersection of arbitrarily many closed sets is closed, and so all we need to do is show that $\mathbb{V}(f)$ is closed for any $f \in \mathbb{C}[X_1, \ldots, X_n]$.

We have

$$\mathbb{V}(f) = \{z \in \mathbb{C}^n \mid f(z) = 0\},$$

or, equivalently,

$$\mathbb{V}(f) = f^{-1}(\{0\}).$$

Then since polynomial maps are continuous (in the topological sense), the preimage of the closed singleton set $\{0\}$ must also be closed. □

Corollary 4. $\mathrm{GL}_2(\mathbb{C})$ *cannot be viewed as a variety in $\mathbb{C}^4$.*

Proof. Note that $\mathrm{GL}_2(\mathbb{C})$ is the complement of the set

$$\mathbb{V}(X_1X_4 - X_2X_3) \subset \mathbb{C}^4.$$

This *is* a variety, and is hence closed in $\mathbb{C}^4$, meaning that $\mathrm{GL}_2(\mathbb{C})$ is open. Then, since the only clopen sets in $\mathbb{C}^4$ are $\mathbb{C}^4$ and $\emptyset$, it cannot also be closed, and therefore cannot be a variety. □

Next lecture, we will discuss *morphisms* (structure-preserving maps) between varieties.