Topics in modern geometry

Exercise sheet 2

Exercise 1.
As in the proof of Hilbert's basis theorem, let R be a Noetherian ring and let I be an ideal of $R[X]$. For each n we define

$$I_n = \{a_n \in R \mid \exists f = a_n X^n + a_{n-1} X^{n-1} + \cdots + a_0 \in I\}.$$

Show that each I_n is an ideal of R and that $I_n \subseteq I_{n+1}$.

Exercise 2.
Consider the ring $\mathbb{R}[X, Y]$ and the polynomial $f = X^2 + Y^2$. Describe the variety $\mathbb{V}(f)$ and the ideal $\mathbb{I}(\mathbb{V}(f))$.

Exercise 3.
Consider the polynomial $f = X^2 + 1$. We have seen that in the reals $\mathbb{V}(f) = \varnothing$ so that $(f) \subsetneq \mathbb{I}(\mathbb{V}(f))$. Show that as a polynomial in $\mathbb{C}[X]$. We have the equality $(f) = \mathbb{I}(\mathbb{V}(f))$.

Exercise 4.
Consider the ideal $J = (X^2 + Y^2 + Z^2, XY + XZ + YZ) \subseteq \mathbb{C}[X, Y, Z]$. Describe the variety $\mathbb{V}(J)$ and show that $\mathbb{I}(\mathbb{V}(J)) \neq J$.

Exercise 5.
Let $V = V_1 \cup V_2 \cup \cdots \cup V_s = W_1 \cup W_2 \cup \cdots \cup W_t$ be two decompositions of V into unions of irreducible varieties with $V_i \not\subset V_j$ and $W_i \not\subset W_j$ for all $i \neq j$ (as in proposition 17). Prove that $s = t$ and that the W_i are simply a permutation of the V_i.

Exercise 6.
Let k be an algebraically closed field and let $K \cong k$. Show that any homomorphism $\varphi\colon k \to K$ must be an isomorphism.

Hint: Recall that a homomorphism of fields is necessarily injective. It may be helpful to consider the image of k, $\overline{k} = \varphi(k)$, as a subfield of K.

Exercise 7.

1. Prove that a prime ideal is radical.

2. Prove that a maximal ideal is prime.

Exercise 8.

Let I be and ideal of some ring R. We say that I is a *primary ideal* if whenever $fg \in I$ then either $f \in I$ or $g^m \in I$ for some positive integer m.

Prove that if I is a primary ideal then $\operatorname{rad} I$ is prime.

Exercise 9.

Let I and J be ideals of the polynomial ring $k[X_1, \ldots, X_n]$ over some field k.

1. Show that $IJ \subseteq I \cap J$.

2. Show that $\mathbb{V}(I) \cup \mathbb{V}(J) = \mathbb{V}(IJ)$.

3. Show that $\mathbb{V}(I) \cup \mathbb{V}(J) = \mathbb{V}(I \cap J)$.

4. Give an example of ideals I and J where $IJ \neq I \cap J$.

Note that the product of ideals is given by: $IJ = \{fg \,|\, f \in I \text{ and } g \in J\}$.