THE JCR LECTURE SERIES

SIMON PEACOCK

Abstract. These notes were collected during an adhoc series of seminars presented by Jeremy Rickard in 2011 and 2012 under the title *Things that might be useful to know during your PhD*. The topics covered include quivers and path algebras; simple modules, indecomposable projectives and their correspondence; the functor category; and Auslander–Reiten Theory.

1. Notation and conventions

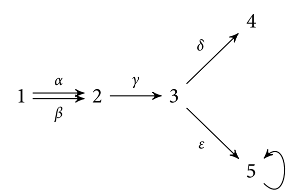
- k will always denote an algebraically closed field;
- A will always denote a finite dimensional k-algebra;
- The notation ${}_RL$, M_S, ${}_RN_S$ denotes that L is a left R-module, M is a right S-module and N is a left R–right S-bimodule;
- Functors will be k-additive.
- The material can be extended to the representation theory of Artin algebras, for more information see [ARS95].

[ARS95] Auslander, Reiten, and Smalø, *Representation theory of Artin algebras*, Cambridge Studies in Advanced Mathematics, vol. 36, Cambridge University Press, Cambridge, 1995

2. Fundamentals

Definition (Quiver)**.** A *quiver* $Q = (V, E)$ is a directed graph. We define the maps $s\colon E \to V$ and $t\colon E \to V$ to be the source and target maps. If $\alpha \in E$ is an edge from $e_1 \in V$ to $e_2 \in V$ then $s(\alpha) = e_1$ and $t(\alpha) = e_2$.

Example.



Definition (Path)**.** A *path* in a quiver Q is either sequence of edges $p = \alpha_1 \dots \alpha_n$ with $t(\alpha_i) = s(\alpha_{i+1})$ for each $1 \le i < n$ or a vertex $e_i \in V$. The paths $e_i \in V$ are called *trivial paths*. The definitions of s and t are extended in the obvious way for paths.

Definition (Path algebra)**.** The *path algebra* for a quiver Q over a field k, denoted kQ is the k-vector space with basis the set of all paths. For $p, q \in kQ$ the multiplication in kQ is defined as follows

$$pq = \begin{cases} \alpha_1 \dots \alpha_m \beta_1 \dots \beta_n & \text{if } p = \alpha_1 \dots \alpha_m \\ & \quad q = \beta_1 \dots \beta_n \text{ and} \\ & \quad t(\alpha_m) = s(\beta_1) \\ p & \text{if } q = e_{t(p)} \\ q & \text{if } p = e_{s(q)} \\ 0 & \text{otherwise} \end{cases}$$

Note that the identity element of kQ is $1 = \sum_{i \in V} e_i$ a sum of orthogonal idempotents.

Definition (Representation)**.** A *representation* of a quiver Q over a field k is a pair (V, f) with V a set of vector spaces $\{ V_i \mid i \text{ a vertex of } Q \}$ and f a set of k-linear maps $\left\{ f_\alpha\colon V_i \to V_j \,\middle|\, i \xrightarrow{\alpha} j \text{ an edge of } Q \right\}$. A representation is finite dimensional if each V_i is finite dimensional.

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A morphism of representations $h\colon (V, f) \to (W, g)$ is a set of k-linear maps $h_i\colon V_i \to W_i$ such that for each $i \xrightarrow{\alpha} j$ the following square commutes

$$\begin{array}{ccc} V_i & \xrightarrow{h_i} & W_i \\ {\scriptstyle f_\alpha}\downarrow & & \downarrow{\scriptstyle g_\alpha} \\ V_j & \xrightarrow{h_j} & W_j \end{array}$$

Given a kQ-module, M, we can obtain a representation of the quiver Q over k. Since $1 = \sum_i e_i \in kQ$, we have that $M = \bigoplus_i Me_i$, a direct sum of vector spaces, one for each vertex. We also have for each edge $i \xrightarrow{\alpha} j \in Q$ that $\alpha = e_i \alpha e_j$ and hence α induces a map $\mu_\alpha\colon Me_i \to Me_j$ of multiplication by α. Also from a kQ-homomorphism, $h\colon M \to M'$ we get a morphism of representations in the obvious way:

$$h_i\colon Me_i \to M'e_i$$
$$me_i \mapsto h(me_i) = h(m)e_i$$

and commutativity of the square is given by $\mu'_\alpha h(me_i) = h(me_i)\alpha = h(me_i\alpha) = h(\mu_\alpha(me_i)) = h\mu_\alpha(me_i)$.

Conversely, given a representation (V_i, f_α) we can construct a module, M, for kQ. We define $M = \bigoplus_i V_i$ as a vector space with the action generated by

$$\begin{array}{rcl} kQ & \longrightarrow & \mathrm{End}(M) \\ e_i & \mapsto & \iota_i \pi_i \\ \alpha_1 \ldots \alpha_n & \mapsto & \iota_{t(\alpha_n)} f_{\alpha_n} \ldots f_{\alpha_1} \pi_{s(\alpha_1)} \end{array}$$

where $\iota_i\colon V_i \to M$ is the natural inclusion and $\pi_i\colon M \to V_i$ is the natural projection. Again a morphism of representations, h_i gives rise to a kQ-homomorphism, $\bigoplus_i h_i$ of the constructed modules. Under this correspondence we have the following proposition.

Proposition 2.1. *The category* $\operatorname{mod} kQ$ *of finitely generated kQ-modules and* $\operatorname{rep}_k Q$ *of finite dimensional representations of Q over k are equivalent.*

Definition (Quiver with relations)**.** A *relation* on a quiver Q is a k-linear sum of paths from a vertex i to a vertex j. That is $\sigma = \sum_n a_n p_n \in kQ$ with $a_n \in k$ and $i = s(p_n)$ and $j = t(p_n)$ for all n. If $\rho = \{p_t\}$ is a set of relations then the pair (Q, ρ) is a *quiver with relations* and its associated algebra is the quotient $kQ/\langle\rho\rangle$. We will mainly consider relations for which $J^t \leq \langle\rho\rangle \leq J^2$, where J is the ideal generated by the arrows of Q. When this is the case we say ρ is a set of admissible relations.

Proposition 2.2 (Semisimple)**.** *The following are equivalent for a finite dimensional algebra A:*

- *A is semisimple;*
- *The regular module, A_A, is semisimple;*
- *All A-modules are semisimple.*

Note. The endomorphism ring $\mathrm{End}(A_A)$ is isomorphic to A under the mapping $\phi \mapsto \phi(1)$ and if $A = S_1^{d_1} \oplus \cdots \oplus S_n^{d_n}$ is semisimple then

$$\begin{aligned} \mathrm{End}(A) &= \left\{ \left(\begin{smallmatrix} X_1 & 0 & \cdots & 0 \\ 0 & X_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & X_n \end{smallmatrix} \right) \middle| \, X_i \text{ a } d_i \times d_i \text{ block} \right\} \\ &= \mathrm{M}_{d_1}(k) \times \cdots \times \mathrm{M}_{d_n}(k) \end{aligned}$$

Definition (Radical)**.** Let M_A be a right A-module. The *radical* of M is given by the following equivalent definitions.

$$\begin{aligned} \operatorname{rad} M &= \bigcap \{\ker \phi \mid \phi\colon M \to S,\ S \text{ simple}\} \\ &= \bigcap \{N \mid N \le M,\ N \text{ maximal}\} \\ &= \sum \{N \mid N \le M,\ N \text{ superfluous}\} \end{aligned}$$

Proposition 2.3. *Some facts about the radical for an algebra A and a module M:*

(a) $\operatorname{rad} A$ *is an ideal of A;*
(b) The quotient $\frac{A}{\operatorname{rad} A}$ *is semisimple;*
(c) The quotient $\frac{M}{\operatorname{rad} M} = \operatorname{hd} M$ *is the maximal semisimple quotient of M (called the head of M);*
(d) $\operatorname{rad} M = M \operatorname{rad} A$*;*
(e) $(\operatorname{rad} A)^n = \operatorname{rad}(\operatorname{rad}(\cdots \operatorname{rad} A)\cdots) = 0$ *for some integer n;*
(f) $\operatorname{rad} A$ *is the unique maximal nilpotent ideal.*
(g) If $K \le A$ *is a submodule such that K is nilpotent and* $\frac{A}{K}$ *is semisimple, then* $K = \operatorname{rad} A$.

Consider the module $J \le kQ$ generated by the arrows of Q. It is clear that J is nilpotent and since $kQ/J \cong \bigoplus_i e_i k$ is semisimple, we have that J is the radical of kQ.

Definition (Socle)**.** The *socle*, $\operatorname{soc} M$, of M is dual to the radical and is given by the following equivalent definitions.

$$\begin{aligned} \operatorname{soc} M &= \sum \{\operatorname{im} \phi \mid \phi\colon M \to S,\ S \text{ simple}\} \\ &= \sum \{N \mid N \le M,\ N \text{ simple}\} \\ &= \bigcap \{N \mid N \le M,\ N \text{ essential}\} \end{aligned}$$

Note that there is a correspondence $\operatorname{soc} M = \operatorname{hd}(M^*)^*$.

For a module M and $\phi \in \operatorname{End} M$, we have a chain $M \ge \operatorname{im} \phi \ge \operatorname{im} \phi^2 \ge \cdots$. If M is finite dimensional (which is a standing assumption) then for some k, $\operatorname{im} \phi^k = \operatorname{im} \phi^{k+1}$. In particular ϕ^k is idempotent and $M = \operatorname{im} \phi^k \oplus \ker \phi^k$. We have the following proposition.

Proposition 2.4. *For an indecomposable module M and* $\phi \in \operatorname{End}(M)$ *either* ϕ *is nilpotent or* ϕ *is an isomorphism. As* $\operatorname{rad} \operatorname{End}_A(M)$ *is the unique maximal nilpotent ideal* $\operatorname{rad} \operatorname{End}_A(M) = \{\phi\colon M \to M \mid \phi \text{ is nilpotent}\}$.

Recall that for bimodules ${}_R M_S$ and ${}_R N_T$, we can give $\operatorname{Hom}_R(M, N)$ a left S–right T-bimodule structure via $(sft)(m) = f(ms)t$. Similarly for modules ${}_S M_R$ and ${}_T N_R$ we have a bimodule ${}_T \operatorname{Hom}_R(M, N)_S$ by $(tfs)(m) = tf(sm)$. Note that for R-morphisms the R action is lost; the action for the domain module moves side; and the action for the codomain remains on the same side.

Two important specialisations of the above theory are that for an A-module M_A and a k-vector space V both $\operatorname{Hom}_k(M, V)$ and $\operatorname{Hom}_A(M, A)$ are left A-modules but note the difference in action: if $f \in \operatorname{Hom}_k(M, V)$ and $g \in \operatorname{Hom}_A(M, A)$ then

$$\begin{aligned} (af)(m) &= f(ma) \qquad \text{but} \\ (ag)(m) &= ag(m) \end{aligned}$$

Recall also that for a left A-module N and a right module M, the usual tensor product $M \underset{A}{\otimes} N$ is a vector space over k and the functors

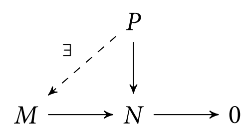
$$(- \underset{A}{\otimes} N)\colon \operatorname{mod} A \to \operatorname{mod} k$$
$$\operatorname{Hom}_k(N, -)\colon \operatorname{mod} k \to \operatorname{mod} A$$

are an adjoint pair so that $\operatorname{Hom}_k(M \underset{A}{\otimes} N, k) \cong \operatorname{Hom}_A(M, \operatorname{Hom}_k(N,k)) \cong \operatorname{Hom}_A(M, N^*)$. That is $M \underset{A}{\otimes} N \cong \operatorname{Hom}_A(M, N^*)^*$.

3. Projectives

Definition (Projective module)**.** A module P_A is called *projective* if equivalently:

- The functor $\operatorname{Hom}_A(P, -)$ is exact;
- For any surjective map $\phi\colon M \to N$ and any map $f\colon P \to N$ there exists $f'\colon P \to M$ such that $f = \phi f'$;



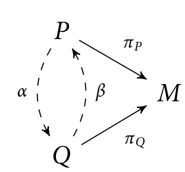
- Any exact sequence $M \to P \to 0$ splits;
- P is a direct summand of A_A^k for some k.

Definition (Projective cover)**.** A *projective cover* of an A-module M is a projective A-module $P = P_M$ of minimal dimension together with a surjection $P \to M \to 0$.

Theorem 3.1: *Uniqueness of projective covers*

If $P \overset{\pi_P}{\to} M \to 0$ is a projective cover and $Q \overset{\pi_Q}{\to} M \to 0$ is any projective module mapping onto M then $Q = P \oplus P'$ for some $P' \leq \ker \pi_Q$. In particular projective covers are unique up to isomorphism.

Proof. As P and Q are projective, each of the maps π_P, and π_Q factors through the other



The composition $\beta\alpha \in \operatorname{End} P$ and so for large k, we have that $P = \operatorname{im}(\beta\alpha)^k \oplus \ker(\beta\alpha)^k$. Now $\ker(\beta\alpha)^k \leq \ker \pi_P(\beta\alpha)^k = \ker \pi_P$ as $\pi_P\beta\alpha = \pi_P$. This shows that $\operatorname{im}(\beta\alpha)^k$ maps onto M and since it is also projective, the minimality of P implies $\beta\alpha$ is an isomorphism.

A similar argument shows that $Q = \operatorname{im}(\alpha\beta)^k \oplus \ker(\alpha\beta)^k \cong P \oplus \ker(\alpha\beta)^k$ with $\ker(\alpha\beta)^k \leq \pi_Q$. □

Proposition 3.2 (Facts about projective covers)**.**

(a) For a simple A-module, S, its projective cover P_S is indecomposable.
(b) A module and its head share a projective cover: $P_M = P_{\operatorname{hd} M}$.
(c) The projective cover of a direct sum is the direct sum of projective covers: $P_{M \oplus N} = P_M \oplus P_N$.

(d) A simple module is isomorphic to the head of its projective cover: $S \cong \operatorname{hd} P_S$.

Proof.

(a) If $P_S = P_1 \oplus P_2$ then each of P_i maps onto 0 or S. The minimality of P_S shows that only one can map onto S and minimality again shows the other must be the zero module.

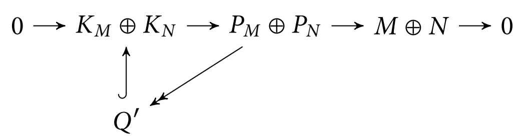
(b) We have the exact sequence $0 \to \operatorname{rad} M \to M \to \operatorname{hd} M \to 0$ and so the projective cover of $P_{\operatorname{hd} M} \to \operatorname{hd} M$ factors through $P_{\operatorname{hd} M} \xrightarrow{\alpha} M$. As $P_{\operatorname{hd} M}$ maps onto the head, $M = \operatorname{im} \alpha + \operatorname{rad} M$ and since for finite dimensional modules the radical is superfluous $M = \operatorname{im} \alpha$. Minimality of the projective covers now shows the result.

(c) By the uniqueness of projective covers for some projective module Q we have

$$P_{M \oplus N} \oplus Q = P_M \oplus P_N \xrightarrow{\phi} M \oplus N \to 0$$

with $Q \leq \ker \phi$.

Let K_M and K_N be the kernels of the projective covers and $Q' \leq Q$ an indecomposable summand so that we have the following diagram.



and so we have $\alpha_M : Q' \to K_M \to P_M \to Q'$ and $\alpha_N : Q' \to K_N \to P_N \to Q'$ with $\alpha_i \in \operatorname{End} Q'$. As Q' is indecomposable α_i is either nilpotent or an isomorphism, but since $\alpha_M + \alpha_N = 1_{Q'}$ both cannot be nilpotent. By minimality we now have that Q' is zero and the result follows.

(d) The head of a module is semisimple so let $\operatorname{hd} P_S = S_1 \oplus \cdots \oplus S_n$. Now $P_S \twoheadrightarrow \operatorname{hd} P_S$ and so by uniqueness of projective covers and part (c), $P_{S_1} \oplus \cdots \oplus P_{S_n}$ is a summand of P_S. Now part (a) gives the result.

□

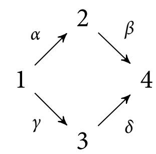
The above proposition gives rise to a one-to-one correspondence between simple modules and indecomposable projectives (up to isomorphism)

$$\left\{\begin{matrix}\text{simple}\\ \text{modules}\end{matrix}\right\} \underset{\operatorname{hd} P \,\leftarrow\!\shortmid\, P}{\overset{S \mapsto P_S}{\rightleftarrows}} \left\{\begin{matrix}\text{indecomposable}\\ \text{projective}\\ \text{modules}\end{matrix}\right\}$$

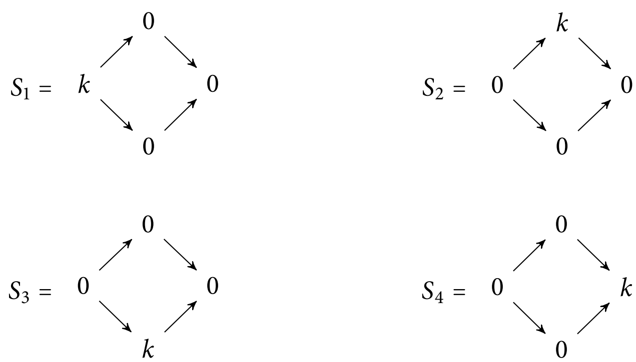
and a similar correspondence for injective modules:

$$\left\{\begin{matrix}\text{simple}\\ \text{modules}\end{matrix}\right\} \underset{\operatorname{soc} I \,\leftarrow\!\shortmid\, I}{\overset{S \mapsto (P_{S^*})^*}{\rightleftarrows}} \left\{\begin{matrix}\text{indecomposable}\\ \text{injective}\\ \text{modules}\end{matrix}\right\}$$

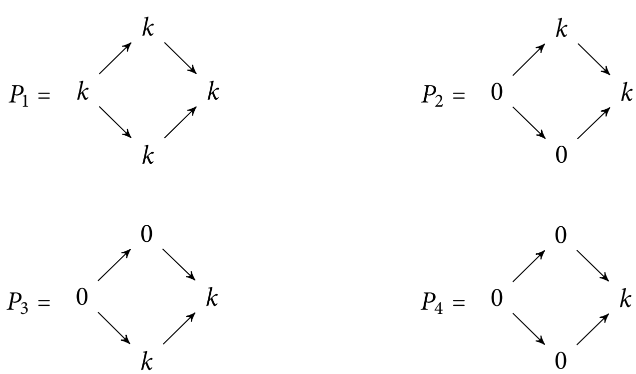
Example. Consider the following quiver with the relation $\alpha\beta = \gamma\delta$.



The simple modules are



Using the fact that $1 = \sum_i e_i$ we can decompose the regular module $kQ = \bigoplus_i e_i kQ$ and so $P_i = e_i kQ = \langle p \mid s(p) = i \rangle$ is projective. Since $P_i / \operatorname{rad} P_i = e_i k$ is simple we see that P_i are the indecomposable projectives.



In a similar fashion $I_i = \langle p \mid t(p) = i \rangle$, see the discussion on page 10.

$$I_1 = \begin{array}{ccccc} & & 0 & & \\ & \nearrow & & \searrow & \\ k & & & & 0 \\ & \searrow & & \nearrow & \\ & & 0 & & \end{array} \qquad I_2 = \begin{array}{ccccc} & & k & & \\ & \nearrow & & \searrow & \\ k & & & & 0 \\ & \searrow & & \nearrow & \\ & & 0 & & \end{array}$$

$$I_3 = \begin{array}{ccccc} & & 0 & & \\ & \nearrow & & \searrow & \\ k & & & & 0 \\ & \searrow & & \nearrow & \\ & & k & & \end{array} \qquad I_4 = \begin{array}{ccccc} & & k & & \\ & \nearrow & & \searrow & \\ k & & & & k \\ & \searrow & & \nearrow & \\ & & k & & \end{array}$$

Theorem 3.3:

> The number of times a simple module S occurs in a composition series of a module M is $\dim_k \mathrm{Hom}_A(P_S, M)$.

Proof. We prove this by induction on the dimension of M.

Firstly if M is simple then $\mathrm{Hom}(P_S, S) \cong k$ and for a simple module $T \not\cong S$, $\mathrm{Hom}(P_T, S) = 0$.

Now assume the theorem is true for modules of dimension less than $\dim M$.

Let $T \leq M$ be a simple submodule, and let $M' = M/T$. We have an exact sequence

$$0 \to T \to M \to M' \to 0$$

and after applying the functor $\mathrm{Hom}_A(P_S, -)$ we obtain

$$0 \to \mathrm{Hom}_A(P_S, T) \to \mathrm{Hom}_A(P_S, M) \to \mathrm{Hom}_A(P_S, M') \to 0.$$

The number of times S occurs in a composition series of M is the number of times S occurs in a composition series of M' if $S \not\cong T$ and it is one greater if $S \cong T$. The result now follows immediately. □

Corollary 3.4 (Jordan-Hölder theorem)**.** *Any two composition series of a module M are equivalent.*

Theorem 3.5:

> Let $\mathcal{P}_A$ denote the category of finitely generated projective A-modules.
>
> The categories $\mathrm{mod}\, A$ and the functor category $\mathrm{Fun}(\mathcal{P}_A^{\mathrm{op}}, \mathrm{mod}\, k)$ are equivalent under the mappings
>
> $$\begin{array}{rcl} \mathrm{mod}\, A & \leftrightarrow & \mathrm{Fun}(\mathcal{P}_A^{\mathrm{op}}, \mathrm{mod}\, k) \\ M & \mapsto & \mathrm{Hom}_A(-, M) \\ F(A) & \mapsfrom & F \end{array}$$

Proof.

$\circlearrowright$: $\mathrm{Hom}_A(A, M) \cong M$.

$\circlearrowleft$: By additivity of the functors we need only check for the regular module, but $\mathrm{Hom}_A(A, FA) \cong FA$. □

Crazy Propaganda [A] (Finitely generated module). Given the above theorem we have an alternative definition: a finitely generated A-module over a field k is a functor from the opposite category of finitely generated projective A-modules to the category of k–vector spaces.

For an A-module M, let $\operatorname{add} M = \{N \mid N \text{ is a summand of } M^k \text{ for some } k\}$, so that $\operatorname{add} A = \mathcal{P}_A$.

Let $E = \operatorname{End}_A(M)$ so that $\operatorname{Hom}_A(M,-)\colon \operatorname{mod} A \to \operatorname{mod} E$ is a functor from A-modules to E-modules. It is clear that this functor takes $M_A \mapsto E_E$ and restricting to $\operatorname{add} M$ we have an equivalence of categories $\operatorname{add} M \xrightarrow{\sim} \mathcal{P}_E$.

Let $P_1, \dots, P_n$ be a complete list of indecomposable projective A-modules and $Q = P_1^{d_1} \oplus \cdots \oplus P_n^{d_n}$ with $d_i > 0$ for all i. By the above argument we have an equivalence $\mathcal{P}_A = \operatorname{add} Q \simeq \mathcal{P}_{\operatorname{End} Q}$ and using theorem 3.5 we have $\operatorname{mod} A \simeq \operatorname{mod} \operatorname{End} Q$.

Conversely, given an equivalence of module categories $\operatorname{mod} A \simeq \operatorname{mod} B$, then there is some A-module X_A such that $X_A \leftrightarrow B_B$ and we must have that $X \cong P_1^{d_1} \oplus \cdots \oplus P_n^{d_n}$ with each $d_i > 0$. Under this equivalence simples modules for A map to simple modules for B and we have

$$\operatorname{Hom}_A(X_A, S_A) = \operatorname{Hom}_B(B_B, S'_B) \cong S'.$$

where the isomorphism is as vector spaces. This demonstrates that the simple module S'_i associated with the projective P'_i in B is d_i-dimensional.

Definition (Basic algebra)**.** An algebra is known as a *basic algebra* if all its simple modules are 1-dimensional.

Theorem 3.6:

Every basic algebra is the path algebra of a finite quiver with admissible relations.

Proof. Suppose E is basic so that $E = \operatorname{End}(P_1 \oplus \cdots \oplus P_n)$ and let e_i be the projection onto P_i. The identity map $1 = \sum_i e_i$, a sum of orthogonal idempotents.

For each i, j consider the spaces $\frac{e_i(\operatorname{rad} E)e_j}{e_i(\operatorname{rad} E)^2 e_j}$ and for each space choose basis elements. Let $\{x_\alpha \mid \alpha \in I\}$ be the set of all such basis elements. Since $\operatorname{rad} E = \bigoplus e_i(\operatorname{rad} E)e_j$ the set $\{x_\alpha\}$ spans $\frac{\operatorname{rad} E}{\operatorname{rad}^2 E}$. We wish to show that $\langle e_i, x_\alpha\rangle$ generates E as an algebra.

We first show that if $T = \langle \alpha \mid \alpha\colon P_i \to P_j,\ \alpha \text{ not an isomorphism}\rangle$ then the quotient $\frac{E}{T}$ is semisimple. In this quotient any map $\alpha\colon P_i \to P_j$ with $i \neq j$ is zero and so we have

$$\frac{E}{T} = \frac{\operatorname{End} P_1}{\operatorname{rad} \operatorname{End} P_1} \oplus \cdots \oplus \frac{\operatorname{End} P_n}{\operatorname{rad} \operatorname{End} P_n} = k^n = \langle e_i \rangle .$$

Note that this is semisimple and since T is clearly nilpotent $T = \operatorname{rad} E$. Thus

$$(*) \qquad \langle e_i, x_\alpha \rangle = \frac{E}{\operatorname{rad} E} + \frac{\operatorname{rad} E}{\operatorname{rad}^2 E} = \frac{E}{\operatorname{rad}^2 E}.$$

Next we show that if $V \leq E$ is a subspace such that $E' = \langle V \rangle$ and $\frac{E'}{\operatorname{rad}^2 E} = \frac{E}{\operatorname{rad}^2 E}$ then $E = E'$.

We show by induction that $\frac{E'}{\operatorname{rad}^k E} = \frac{E}{\operatorname{rad}^k E}$ which is true by assumption for $k = 2$.

Assume $E'/\operatorname{rad}^k E = E/\operatorname{rad}^k E$:

Let $x \in \operatorname{rad}^k E$ so that $x = \sum s_i t_i$ with $s_i \in \operatorname{rad} E$ and $t_i \in \operatorname{rad}^{k-1} E$. Then there are $\overline{s_i} \equiv s_i \pmod{\operatorname{rad}^2 E}$ and $\overline{t_i} \equiv t_i \pmod{\operatorname{rad}^k E}$ with $\overline{s_i}, \overline{t_i} \in E'$. Now

$$x = \sum s_i t_i \equiv \sum \overline{s_i}\,\overline{t_i} \pmod{\operatorname{rad}^{k+1}}$$

with $\sum \overline{s_i}\,\overline{t_i} \in E'$.

We now have by assumption that for $x \in E$ there is $y \in E'$ such that $x - y = s \in \mathrm{rad}^k$. And by the previous argument we have some $u \in E'$ such that $s - u = t \in \mathrm{rad}^{k+1}$, which gives $x \equiv y + u \pmod{\mathrm{rad}^{k+1}}$ which concludes the induction step. Since rad is nilpotent we must have $E = E'$ and by $(*)$, $E = \langle e_i, x_\alpha \rangle$.

Finally, for each x_α there is a unique i and j such that $x_\alpha = e_i x_\alpha e_j$. We form the quiver, Q, on n vertices with edges $\{x_\alpha\}$ where $i \overset{x_\alpha}{\to} j$ for this unique pair i, j.

We have

$$\begin{array}{ccc} kQ & \twoheadrightarrow & E \\ e_i & \mapsto & e_i \\ x_\alpha & \mapsto & x_\alpha \end{array}$$

is a surjective algebra homomorphism with kernel K, $J^k \le K \le J^2$. □

Example. Consider the algebra

$$A = \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} = \begin{array}{c} \begin{pmatrix} * & * & * & * \end{pmatrix} \\ \oplus \\ \begin{pmatrix} 0 & * & * & * \end{pmatrix} \\ \oplus \\ \begin{pmatrix} 0 & 0 & * & * \end{pmatrix} \\ \oplus \\ \begin{pmatrix} 0 & 0 & 0 & * \end{pmatrix} \end{array} = \begin{array}{c} P_1 \\ \oplus \\ P_2 \\ \oplus \\ P_3 \\ \oplus \\ P_4 \end{array}$$

We have $\mathrm{rad}\, P_1 \cong P_2$, $\mathrm{rad}\, P_2 \cong P_3$, $\mathrm{rad}\, P_3 \cong P_4$ and $\mathrm{rad}\, P_4 \cong 0$. We also have that

$$\mathrm{rad}\, A = \begin{pmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathrm{rad}^2 A = \begin{pmatrix} 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

From this we can see that we have three elements x_α:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \text{and} \qquad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and we have the quiver

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4.$$

We now describe an equivalence between the categories of finitely generated projectives and injectives. Recall that the dual functor from right-modules to left-modules, $D\colon (\mathrm{mod}\, A)^{\mathrm{op}} \xrightarrow{\sim} A\,\mathrm{mod}$ taking $M \mapsto M^* = \mathrm{Hom}_k(M, k)$, is a duality of categories. This forms a correspondence between (right) projective A-modules and (left) injectives.

We can also contruct the A-dual functor $-^\vee\colon (\mathrm{mod}\, A)^{\mathrm{op}} \to A\,\mathrm{mod}$ that maps $M \mapsto M^\vee = \mathrm{Hom}_A(M, A)$. This gives an equivalence between the full subcategories of finitely generated (right) projectives and finitely generated (left) projectives.

$$\left\{ \begin{matrix} \text{f.g. projective} \\ \text{right modules} \end{matrix} \right\}^{\mathrm{op}} \rightleftarrows \left\{ \begin{matrix} \text{f.g. projective} \\ \text{left modules} \end{matrix} \right\}$$

Definition (Nakayama functor)**.** The *Nakayama functor* is the composition of the vector space dual and A-dual functors and forms an equivalence between finitely Generated projective A-modules, $\mathcal{P}_A$ and finitely generated injective modules, $\mathcal{I}_A$:

$$\begin{array}{ccl} \mathcal{P}_A & \leftrightarrow & \mathcal{I}_A \\ P_i & \mapsto & I_i = DP_i^\vee = \mathrm{Hom}_A(P,A)^* \end{array}$$

We now look again at the indecomposable injective modules for a path algebra kQ. We claimed earlier that these are the modules $I_i = \langle p \mid t(p) = i \rangle$. Above we have shown that $I_i = DP_i^\vee$ and use these to demonstrate the earlier claim true. We also describe the action of kQ on I_i.

Firstly consider a module map $f\colon P_i \to kQ$, we have that $f(e_i) = f(e_i^2) = f(e_i)e_i$ and so $t(f(e_i)) = i$. Now consider $p = e_i p \in P_i$ we have that $f(p) = f(e_i)p$ and so f is fully determined by its action on e_i. Let p^*, with $t(p) = i$ denote the map $[e_i \mapsto p]$ so that $P_i^\vee = \langle p^* \mid t(p) = i \rangle$. Also let $p^{**}\colon P_i^\vee \to k$ be the map such that $p^{**}(q^*) = \begin{cases} 1 & p = q \\ 0 & \text{otherwise} \end{cases}$ so that the p^{**}, with $t(p) = i$ form a basis for I_i.

We wish to describe the action of A on I_i and so first describe the action on $P_i^\vee$. Let $a \in A$ and p be a path with $t(p) = i$; $(ap^*)(e_i) = ap^*(e_i) = ap$ and hence $ap^* = (ap)^*$, where 0^* is simply the zero map. Now consider q also with $t(q) = i$, we have $(q^{**}a)(p^*) = q^{**}(ap^*) = q^{**}\big((ap)^*\big)$ and hence

$$q^{**}a = \begin{cases} p^{**} & q = ap \\ 0 & \text{otherwise} \end{cases}$$

That is, a path $a \in kQ$ trims paths in I_i from their source. Although in this discussion elements of I_i have been written with a double asterisk we these are clearly not needed and we have $I_i = \langle p \mid t(p) = i \rangle$ as claimed.

4. Symmetric algebras

Definition (Symmetric algebra)**.** A finite dimensional algebra A, is called *symmetric* if the following equivalent properties hold:

(i) There is a linear map $\theta\colon A \to k$ with $\theta(ab) = \theta(ba)$ and $\ker\theta$ contains no nonzero left or right ideals;

(ii) $A \cong A^*$ as A-bimodules;

(iii) For $M \in \operatorname{mod} A$, $P \in P_A$ there is a vector space isomorphism $\mathrm{Hom}_A(M,P) \cong \mathrm{Hom}_A(P,M)^*$ that is functorial in both M and P;

(iv) For $M \in \operatorname{mod} A$ there is an isomorphism of left A-modules $M^* \cong \mathrm{Hom}_A(M,A)$ that is functorial in M.

Proof.

(i) $\Rightarrow$ (ii) Define $f\colon A \to A^*$, by $f(a) = [b \mapsto \theta(ab)]$, then f is a homomorphism of bimodules and has $\ker f = 0$. Since the modules are isomorphic as vector spaces we have that f is an isomorphism.

(ii) $\Rightarrow$ (i) If $f\colon A \to A^*$ is an bimodule isomorphism then $\theta = f(1)\colon A \to k$ has the required property.

(ii) $\Rightarrow$ (iv) For an A-module M, $\mathrm{Hom}_A(M,A) \cong \mathrm{Hom}_A(M,\mathrm{Hom}_k(A,k))$ as $A \cong A^*$ by assumption and so $\mathrm{Hom}_A(M,A) \cong \mathrm{Hom}_k(M \underset{A}{\otimes} A, k) \cong M^*$.

(iv) $\Rightarrow$ (iii) For M and P as in (iii) we have $\mathrm{Hom}_A(M,P) \cong \mathrm{Hom}_A(M,\mathrm{Hom}_k(P^*,k))$ as $P \cong P^{**}$ for finite dimensional P. Then $\mathrm{Hom}_A(M,P) \cong \mathrm{Hom}_k(M \underset{A}{\otimes} P^*, k)$ and so $\mathrm{Hom}_A(M,P)^* \cong M \underset{A}{\otimes} P^* \cong M \otimes \mathrm{Hom}_A(P,A)$ with the last isomorphism given by (iv).

Define a map

$$\begin{aligned} M \otimes \operatorname{Hom}_A(P, A) &\to \operatorname{Hom}_A(P, M) \\ m \otimes \phi &\mapsto [p \mapsto m\phi(p)] \end{aligned}$$

When we consider $P = A$ we have the following

$$\begin{array}{ccccc} & m \otimes \phi & \longmapsto & [p \mapsto m\phi(p)] & \\ m \otimes \mathrm{id}_A & M \otimes \operatorname{Hom}_A(A, A) & \longrightarrow & \operatorname{Hom}_A(A, M) & \theta \\ \uparrow & \wr\uparrow & & \downarrow\wr & \downarrow \\ m & M & & M & \theta(1) \end{array}$$

So that $m \mapsto m \otimes \mathrm{id}_A \mapsto [p \mapsto mp] \mapsto m$ and the top map must be an isomorphism. Now we have $\operatorname{Hom}_A(M, A)^* \cong M \underset{A}{\otimes} \operatorname{Hom}_A(A, A) \cong \operatorname{Hom}_A(A, M)$ and functoriality of the isomorphism $M^* \cong \operatorname{Hom}_A(M, A)$ gives the result for all P.

(iii) $\Rightarrow$ (ii) Let $M = P = A$ then $A \cong \operatorname{Hom}_A(A, A) \cong \operatorname{Hom}_A(A, A)^* \cong A^*$.

□

Note that if $A = kG$ is a group algebra for a finite group G then $\theta(\sum_g \lambda_g g) = \lambda_e$ satisfies condition (i) and thus (finite) group algebras are symmetric.

Note also that since (iii) is a condition purely in terms of the modules of an algebra, if two algebras have equivalent module categories—that is they are Morita equivalent—then one is symmetric if and only if the other is symmetric. If we have $A \cong A_1 \times A_2$ then A is symmetric if and only if A_1 and A_2 are symmetric.

Definition (Block)**.** Let G be a finite group so that $kG = A_1 \oplus \cdots \oplus A_n$ is a direct sum of indecomposable bimodules. The bimodules A_i are unique up to permutation and are called the *blocks* of kG. Additionally, $kG = A_1 \times \cdots \times A_n$ as a product of algebras.

Proof. We begin with $kG = A_1 \oplus \cdots \oplus A_n$ as bimodules so that $1 = \sum_i e_i$. We have that $A_i A_j \subseteq A_i \cap A_j = \{0\}$ for $i \neq j$ and therefore

$$e_i e_j = \begin{cases} e_i & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

This shows that each A_i is an algebra; it is straightforward to check that $kG \cong A_1 \times \cdots \times A_n$ is an algebra isomorphism.

If we let $kG = M \oplus N$ as bimodules then $M = Me_1 \oplus \cdots \oplus Me_n$ and $N = Ne_1 \oplus \cdots \oplus Ne_n$. We must then have $A_i = Me_i \oplus Ne_i$ and since A_i is indecomposable we can define $J = \{i \mid A_i = Me_i\}$ and we have $M = \bigoplus_{i \in J} A_i$, and $N = \bigoplus_{i \notin J} A_i$. □

Note that since group algebras are symmetric, we can see from 4 (ii) that blocks are also symmetric algebras.

If X_{kG} is a kG-module then $X = Xe_1 \oplus \cdots \oplus Xe_n$ with each Xe_i an A_i-module; this demonstrates that each indecomposable kG-module *belongs* to a block. Consider indecomposable modules X and Y, that belong to different blocks so that $X = Xe_i$, $Y = Ye_j$ with $i \neq j$ and let $\phi \colon X \to Y$ be a module homomorphism. Then $\phi(x) = \phi(x)e_j = \phi(xe_j) = 0$ for any x and so there are no non-zero maps between modules of different blocks.

Crazy Propaganda [B]. Let $\mathrm{Id}\colon \operatorname{mod} kG \to \operatorname{mod} kG$ be the identity functor so we have

$$\begin{aligned}\mathrm{Id}(X) &= Xe_1 \oplus \cdots \oplus Xe_n\\ &= F_1(X) \oplus \cdots \oplus F_n(X)\end{aligned}$$

for functors $\{F_i\}$ and so a block is really an indecomposable direct summand of the identity functor.

5. Functor category

As we saw in theorem 3.5 the category $\operatorname{mod} A$ is equivalent to the functor category $\mathrm{Fun}_k(\text{proj–}A^{\mathrm{op}}, \operatorname{mod} k)$ and so we now consider some theory for the larger functor category of $\mathrm{Fun}_k(\operatorname{mod} A^{\mathrm{op}}, \operatorname{mod} k) = \mathrm{Fun}(A)$.

We begin with Yoneda's lemma, which is usually stated in terms of functors to Set, however we are concerned with the k-additive form.

Lemma 5.1 (Yoneda)**.** *Let $\mathcal{C}$ be a pre-k-additive category and $F\colon\mathcal{C} \to \operatorname{mod} k$ a k-additive functor. For $M \in \mathcal{C}$ there is a natural isomorphism of vector spaces*

$$\left\{ \eta\colon \mathrm{Hom}_{\mathcal{C}}(-, M) \to F \;\middle|\; \eta \text{ a natural transformation}\right\} \cong FM$$

$$\eta \mapsto \eta_M(\mathrm{id}_M)$$

Lemma 5.2 (Hom is projective)**.** *The hom functor* $\mathrm{Hom}_A(-, M)$ *is a projective object in* $\mathrm{Fun}(A)$.

Proof. Let $F_1 \xrightarrow{\alpha} F_2 \to 0$ be an exact sequence in $\mathrm{Fun}(A)$ and $\eta\colon \mathrm{Hom}_A(-, M) \to F_2$ a natural transformation. Thus we have an exact sequence

$$\begin{array}{ccccc} x & \longmapsto & \eta_M(\mathrm{id}_M) & & \\ F_1M & \xrightarrow[\alpha_M]{} & F_2M & \longrightarrow & 0\end{array}$$

where $x = \xi_M(\mathrm{id}_M)$ for some natural transformation $\xi\colon \mathrm{Hom}_A(-, M) \to F_1$. The composition $\alpha \circ \xi$ is determined by $(\alpha \circ \xi)_M(\mathrm{id}_M) = \alpha_M(x) = \eta_M(\mathrm{id}_M)$ and so $\alpha \circ \xi = \eta$. □

Definition (Finitely generated)**.** A functor $F \in \mathrm{Fun}(A)$ is *finitely generated* if for some $M \in \operatorname{mod} A$ there is an exact sequence $\mathrm{Hom}_A(-, M) \to F \to 0$.

Definition (Finitely presented)**.** A functor $F \in \mathrm{Fun}(A)$ is *finitely presented* for some $N, M \in \operatorname{mod} A$ there is an exact sequence $\mathrm{Hom}_A(-, N) \to \mathrm{Hom}_A(-, M) \to F \to 0$.

In other words: F is finitely generated and its kernel is also finitely generated.

Note that if $\mathrm{Hom}(-, N) \to \mathrm{Hom}(-, M) \to F \to 0$ is a finite presentation then we obtain $0 \to K \to N \to M$ using Yoneda and taking the kernel K. As Hom is left exact we then get

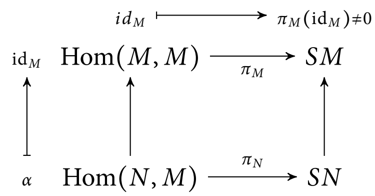
$$0 \to \mathrm{Hom}(-, K) \to \mathrm{Hom}(-, N) \to \mathrm{Hom}(-, M) \to F \to 0.$$

This shows that a finitely presented functor has a projective resolution with three terms.

5.1. **Simple functors.** Let $S \in \mathrm{Fun}(A)$ be a simple functor so that there is some indecomposable module M with $SM \neq 0$. Using Yoneda we must have an exact sequence $\mathrm{Hom}(-, M) \to S \to 0$ showing that S is finitely generated. We now consider the kernel functor K

$$0 \to K \to \mathrm{Hom}(-, M) \xrightarrow{\pi} S \to 0.$$

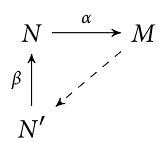
Let $\alpha\colon N \to M$ be a split epimorphism so that $N = M \oplus M'$. The inclusion $M \hookrightarrow M \oplus M' = N$ induces



and hence α is not in the kernel of π_N.

Conversely let us define $\mathrm{rad}_A(N, M) = \{\alpha \in \mathrm{Hom}_A(N, M) \mid \alpha \text{ not a split epimorphism}\}$ and show that $\mathrm{rad}_A(-, M) = \ker \pi$.

To see that $\mathrm{rad}(-, M) < \mathrm{Hom}(-, M)$ is indeed a subfunctor the diagram



shows that if α is not split epic then $\alpha \circ \beta$ cannot be split epic. The fact that $\mathrm{id}_M \notin \mathrm{rad}(M, M)$ shows that it is a proper subfunctor.

We have that $K \le \mathrm{rad}(-, M) < \mathrm{Hom}(-, M)$ and so there exists a θ

$$\theta\colon S \cong \frac{\mathrm{Hom}(-, M)}{K} \twoheadrightarrow \frac{\mathrm{Hom}(-, M)}{\mathrm{rad}(-, M)}$$

As $\theta \ne 0$ we know that $\ker \theta = 0$ and hence $K \cong \mathrm{rad}(-, M)$.

Given the above we can write a simple functor S in the form

$$S^M(N) = \frac{\mathrm{Hom}(N, M)}{\mathrm{rad}(N, M)}$$

with M indecomposable and we have that $S^M(N) = 0$ unless M is a summand of N. In particular $S^M(M) = \frac{\mathrm{End}\, M}{\mathrm{rad}\,\mathrm{End}\, M} \cong k$.

As a special case of the above consider P an indecomposable projective module so that

$$0 \to \mathrm{rad}(-, P) \to \mathrm{Hom}(-, P) \to S^P \to 0$$

is exact.

If $\alpha\colon X \to P$ is a surjection then it necessarily splits. On the otherhand, if α is not a surjection then it must map into $\mathrm{rad}\, P$ as this is the unique maximal submodule. We now have $\mathrm{rad}(-, P) = \mathrm{Hom}_A(-, \mathrm{rad}\, P)$ and that

$$0 \to \mathrm{Hom}(-, \mathrm{rad}\, P) \to \mathrm{Hom}(-, P) \to S^P \to 0$$

is a projective resolution of S^P.

We have shown that in general simple functors S^M are finitely generated and for projective modules S^P is finitely presented. However, the following result shows all simple functors $S \in \mathrm{Fun}(A)$ are finitely presented.

Theorem 5.3: *Auslander-Reiten*

A simple functor S^M is finitely presented.

Proof. Recall that $D = \operatorname{Hom}_k(-,k)$ is the dual functor and $-^\vee = \operatorname{Hom}_A(-,A)$ is the A-dual functor.

Let P be a projective module then

$$\begin{aligned}\operatorname{Hom}_A(-,DP^\vee) &= \operatorname{Hom}_A(-,D\operatorname{Hom}_A(P,A))\\ &\cong \operatorname{Hom}_k(-\underset{A}{\otimes}\operatorname{Hom}_A(P,A),k)\\ &= D(-\underset{A}{\otimes}\operatorname{Hom}_A(P,A))\\ &\cong D\operatorname{Hom}_A(P,-)\end{aligned}$$

with the last isomorphism given by $m\otimes\phi \mapsto [p\mapsto m\phi(p)]$.

For any module X we can construct the map

$$\begin{array}{ccccc}\operatorname{Hom}_A(X,M)\underset{k}{\otimes}\operatorname{Hom}_A(M,X) & \longrightarrow & \operatorname{End} M & \longrightarrow & \dfrac{\operatorname{End} M}{\operatorname{rad}\operatorname{End} M}\cong k\\ \alpha\otimes\beta & \longmapsto & \alpha\circ\beta & & \end{array}$$

Thinking of $\alpha\circ\beta$ as its image in k this gives rise to a natural map

$$\begin{aligned}\operatorname{Hom}_A(X,M) &\to D\operatorname{Hom}_A(M,X)\\ \alpha &\mapsto [\beta\mapsto\alpha\circ\beta].\end{aligned}$$

Hence we have a natural transformation

$$\phi\colon \operatorname{Hom}_A(-,M)\longrightarrow D\operatorname{Hom}_A(M,-).$$

If $\alpha\in\operatorname{rad}(X,M)$, then $\alpha\circ\beta\in\operatorname{rad}\operatorname{End} M$ for all β and so $\ker\phi\le\operatorname{rad}(-,M)$. Conversely if α is a split epimorphism then taking β such that $\alpha\circ\beta=\mathrm{id}_M$ shows that $\ker\phi=\operatorname{rad}(-,M)$.

We now have that $\operatorname{im}\phi\cong S^M$ and so we can factor ϕ via

$$\operatorname{Hom}_A(-,M)\longrightarrow S^M\longrightarrow D\operatorname{Hom}_A(M,-)$$

Now let $P_1\to P_0\to M\to 0$ be a minimal projective presentation (via projective covers) of M. As $D\operatorname{Hom}$ is right exact we obtain

$$\begin{array}{ccccccc} D\operatorname{Hom}_A(P_1,-) & \longrightarrow & D\operatorname{Hom}_A(P_0,0) & \longrightarrow & D\operatorname{Hom}_A(M,-) & \longrightarrow & 0\\ \| & & \| & & \uparrow & & \\ \operatorname{Hom}_A(-,DP_1^\vee) & \longrightarrow & \operatorname{Hom}_A(-,DP_0^\vee) & & & & \\ & & \uparrow{\scriptstyle\psi} & & & & \\ & & \operatorname{Hom}_A(-,M) & \longrightarrow & S^M & \longrightarrow & 0 \end{array}$$

where ψ exists as $\operatorname{Hom}_A(-,M)$ is projective.

Using Yoneda we can move between natural transformations $\operatorname{Hom}(-,X_1)\to\operatorname{Hom}(-,X_2)$ and module homomorphisms $X_1\to X_2$. Thus we can construct the following diagram where Y is the pullback and τM is the kernel of $DP_1^\vee\to DP_0^\vee$ (and hence also the kernel of the pullback)*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \tau M & \longrightarrow & Y & \dashrightarrow & M\\ & & \| & & \downarrow & & \downarrow\\ 0 & \longrightarrow & \tau M & \longrightarrow & DP_1^\vee & \longrightarrow & DP_0^\vee \end{array}$$

*Note that we call τM the Auslander-Reiten translation of M. See [ASS06, chapter IV] for more details.

[ASS06] Assem, Simson, and Skowroński, *Elements of the representation theory of associative algebras. Vol. 1*, London Mathematical Society Student Texts, vol. 65, Cambridge University Press, Cambridge, 2006,

Now by applying the Hom, which is left exact, we can complete the diagram from above

$$\begin{array}{ccccccccc}
 & & & & D\operatorname{Hom}_A(P_1,-) & \longrightarrow & D\operatorname{Hom}_A(P_0,0) & \longrightarrow & \cdots \\
 & & & & \| & & \| & & \\
0 & \longrightarrow & \operatorname{Hom}_A(-,\tau M) & \longrightarrow & \operatorname{Hom}_A(-,DP_1^\vee) & \longrightarrow & \operatorname{Hom}_A(-,DP_0^\vee) & & \\
 & & \| & & \uparrow & & \uparrow{\scriptstyle\psi} & & \\
0 & \longrightarrow & \operatorname{Hom}_A(-,\tau M) & \longrightarrow & \operatorname{Hom}_A(-,Y) & \longrightarrow & \operatorname{Hom}_A(-,M) & \longrightarrow & \cdots
\end{array}$$

and have that S^M is finitely presented. □

Let M be an indecomposable module, that is not projective, so that S^M is finitely presented by

$$0 \longrightarrow \operatorname{Hom}(-,\tau M) \longrightarrow \operatorname{Hom}(-,Y) \longrightarrow \operatorname{Hom}(-,M) \longrightarrow S^M \longrightarrow 0$$

As M is not projective, by applying this sequence to A we have

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Hom}_A(A,\tau M) & \longrightarrow & \operatorname{Hom}_A(A,Y) & \longrightarrow & \operatorname{Hom}_A(A,M) & \longrightarrow & S^M(A) & \longrightarrow & 0 \\
 & & \wr\| & & \wr\| & & \wr\| & & \wr\| & & \\
0 & \longrightarrow & \tau M & \longrightarrow & Y & \longrightarrow & M & \longrightarrow & 0 & &
\end{array}$$

Now consider $\theta \in \operatorname{rad}(X,M)$. By construction $\operatorname{rad}(-,M)$ was the image of $\operatorname{Hom}(-,Y)$ in $\operatorname{Hom}(-,M)$ and hence there is a $\tilde{\theta}$ lifting θ:

$$\begin{array}{ccccccccc}
 & & & & & \overset{\tilde\theta}{\swarrow} & X & & \\
 & & & & & & \downarrow{\scriptstyle\theta} & & \\
0 & \longrightarrow & \tau M & \longrightarrow & Y & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

Notice for that the same reason if θ is a split epimorphism then no such $\tilde{\theta}$ exists. Such a sequence is called almost split (see the following definitions).

5.2. **Almost split sequences.**

Definition (Almost split)**.** A map $\alpha\colon N \to M$ is called *right almost split* if the following two conditions are satisfied

- α is not a split epimorphism;
- if $\theta\colon X \to M$ is not a split epimorphism then there exists $\tilde{\theta}\colon X \to N$ such that $\alpha\tilde{\theta} = \theta$.

If additionally $\ker\alpha$ contains no (non-zero) summands of N then α is called *minimal right almost split*. The definition of (minimal) left almost split is dual to that above.

Definition (Almost split sequence)**.** An exact sequence $0 \to K \to N \xrightarrow{\alpha} M \to 0$ is called an *almost split sequence* if α is minimal right almost split.

Note that by proposition 5.5 we can see the definition could easily have been in terms of minimal left almost split maps.

Theorem 5.4: *Uniqueness of minimal almost split*

> If $X \xrightarrow{\alpha} M$ is minimal right almost split and $Y \xrightarrow{\beta} M$ is right almost split (not necessarily minimal), then $Y \cong X \oplus X'$ for some $X' \leq \ker \beta$. In particular minimal right almost split maps are unique up to isomorphism.

Proof. The proof follows a similar argument to the uniqueness of projective covers, theorem 3.1. □

Proposition 5.5. *The property of being an almost split sequence is self dual. That is if* $0 \to K \xrightarrow{\beta} N \xrightarrow{\alpha} M \to 0$ *is an almost split sequence then* β *is minimal left almost split.*

Proof. Assume that α is minimal right almost split and $\phi\colon K \to X$ not a split monomorphism. If N' is the pushout of $K \to N$ and $K \to X$ then we have the diagram below.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{\beta} & N & \xrightarrow{\alpha} & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\theta} & & \| & & \\ 0 & \longrightarrow & X & \xrightarrow{f} & N' & \xrightarrow{g} & M & \longrightarrow & 0 \end{array}$$

If g is a split epimorphism then f is split monic and ϕ trivially factors through $\theta\beta$.

Now if g is not a split epimorphism then g is right almost split since any not split epic map to M factors through α and hence through g. By the uniqueness of minimal right almost split maps we have $N' = N \oplus N''$ and $N'' \leq \ker g$. Now $X = \ker g = \ker \alpha \oplus N'' = K \oplus N''$, which contradicts the assumption. □

It is fairly straightforward to see that the last map in the sequence $0 \to \tau M \to Y \to M \to 0$ from theorem 5.3 is minimal right almost split and so the sequence is almost split. Also by the following lemma τM is also indecomposable.

Lemma 5.6. *(a) If* $\alpha\colon M \to N$ *is minimal right almost split then* N *is indecomposable.*
(b) If $\alpha\colon M \to N$ *is minimal left almost split then* M *is indecomposable.*

Proof.

(a) Assume that $N = N_1 \oplus N_2$ with $N_1 \neq 0 \neq N_2$. Then the inclusion $\xi_i\colon N_i \to N$ is not a split epimorphism and so there is $\pi_i\colon N_i \to M$ with $\alpha\pi_i = \xi_i$. Clearly $1_N = \xi_1 + \xi_2 = \alpha(\pi_1 + \pi_2)$ and α is split epic.

□

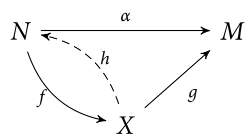
Definition (Irreducible)**.** A map $\theta\colon X \to Y$ is called *irreducible* if it is neither split epic nor split monic and, if the diagram $\begin{array}{ccc} X & \xrightarrow{\theta} & Y \\ & {\scriptstyle f}\searrow \quad \nearrow{\scriptstyle g} & \\ & Z & \end{array}$ commutes then either f is split monic or g is a split epic.

Lemma 5.7. *Any irreducible map* $\theta\colon X \to Y$ *is either an epimorphism or a monomorphism.*

Proof. We have $\begin{array}{ccc} X & \xrightarrow{\theta} & Y \\ & \searrow \quad \nearrow & \\ & \operatorname{im}\theta & \end{array}$ and so $\operatorname{im}\theta$ is isomorphic to either X or Y. □

Proposition 5.8. *If $\alpha\colon N \to M$ is minimal right almost split the α is irreducible.*

Proof. Assume for some X we have the following diagram



Now if g is not split epic then there is an h such that $\alpha h = g$. We must have hf is an isomorphism by minimality of α and so f is split monic. □

We have previously define $\operatorname{rad}(-, M)$ for indecomposable modules M, we now generalise the definition.

Definition ($\operatorname{rad}(X,Y)$)**.** Let $X, Y \in \operatorname{mod} A$. We define $\operatorname{rad}(X,Y) \leq \operatorname{Hom}_A(X,Y)$ to be the set of maps that do not map any summand of X isomorphically to a summand of Y. That is

$$\begin{aligned}\operatorname{rad}(X,Y) \\ = \Big\{ f\colon X \to Y \,\Big|\, &\text{if } \alpha \text{ is the composition} \\ &M \to X \xrightarrow{f} Y \to M \text{ with } M \text{ indecomposable,} \\ &\text{then } \alpha \text{ is not an isomorphism} \Big\}\end{aligned}$$

We define inductively $\operatorname{rad}^n(X,Y)$ as

$$\begin{aligned}\operatorname{rad}^n(X,Y) \\ = \Big\{ f\colon X \to Y \,\Big|\, &f = gh, \text{ with } g \in \operatorname{rad}^{n-1}(Z,Y) \\ &\text{and } h \in \operatorname{rad}(X,Z) \text{ for some } Z \Big\} \\ = \Big\{ f\colon X \to Y \,\Big|\, &f = gh, \text{ with } g \in \operatorname{rad}^{n-k}(Z,Y) \\ &\text{and } h \in \operatorname{rad}^k(X,Z) \text{ for some } Z, 0 < k < n \Big\}\end{aligned}$$

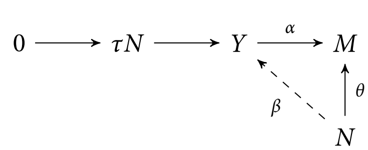
Note that for indecomposable Y this matches the earlier definition and also if $\alpha \in \operatorname{rad}(X,Y)$ then $S^M(\alpha) = 0$ for all indecomposable modules M.

If M and N are indecomposable then we have

$$\operatorname{rad}(N,M) = \begin{cases} \operatorname{Hom}_A(N,M) & M \not\cong N \\ \operatorname{rad}\operatorname{End} M & M \cong N \end{cases}$$

For M, N indecomposable it is clear to see that $\{\alpha\colon N \to M \,|\, \alpha \text{ irreducible}\} = \operatorname{rad}(N,M) \setminus \operatorname{rad}^2(N,M)$.

Take again the sequence $0 \to \tau M \to Y \xrightarrow{\alpha} M$ and consider an irreducible map $\theta\colon N \to M$ from an indecomposable module N to M.



Since θ is not split epic it must factor through α and so β must be split monic. This means that N is a summand of Y and since (Krull-Schmidt) there are only finitely many summands we know that there are only finitely many N for which an irreducible map exists.

Now for indecomposable modules M and N, consider the space $\frac{\mathrm{rad}(N,M)}{\mathrm{rad}^2(N,M)}$ with basis $\{\alpha_1, \ldots, \alpha_n\}$. Since this is a basis we clearly have $\sum \alpha_i \notin \mathrm{rad}^2(N, M)$ and hence irreducible, this immediately gives $Y = N^n \oplus Y'$. Conversely, if $Y = N^n \oplus Y'$ and $\alpha\colon Y \to M$ is minimal right almost split (and therefore irreducible) then $\alpha = \alpha_1 \oplus \cdots \oplus \alpha_n \oplus \beta$ where $\alpha_i\colon N \to M$ and $\beta\colon Y' \to M$, then we have any non-zero linear combination of these is irreducible and hence $\dim\left(\frac{\mathrm{rad}(N,M)}{\mathrm{rad}^2(N,M)}\right) \geq n$. This shows that the following definition is consistent.

Definition (Auslander-Reiten quiver)**.** The *Auslander-Reiten* quiver for an algebra A has a vertex for each isomorphism class of an indecomposable module of A. If M and N are indecomposable modules then there are exactly n edges $[N] \to [M]$ where n is equivalently given by

- $n = \dim\left(\frac{\mathrm{rad}(N,M)}{\mathrm{rad}^2(N,M)}\right)$;
- $0 \to \tau M \to N^n \oplus Y \to M \to 0$ is an almost split sequence with N not a summand of Y.
- $0 \to N \to M^n \oplus X \to \tau^{-1} N \to 0$ is an almost split sequence with M not a summand of X;

References

[ASS06] Ibrahim Assem, Daniel Simson, and Andrzej Skowroński, *Elements of the representation theory of associative algebras. Vol. 1*, London Mathematical Society Student Texts, vol. 65, Cambridge University Press, Cambridge, 2006, .

[ARS95] Maurice Auslander, Idun Reiten, and Sverre O. Smalø, *Representation theory of Artin algebras*, Cambridge Studies in Advanced Mathematics, vol. 36, Cambridge University Press, Cambridge, .