Diagram chasing with lizards

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1 Introduction and definitions

Definition (Chain complex)**.** *A* ***chain complex*** (C, d) *of abelian groups is a sequence*

$$\cdots \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} C_{-1} \xrightarrow{d_{-1}} \cdots$$

such that $d_i \circ d_{i+1} = 0$ *for all* i*.*

In such a case the image of d_{i+1} is a subgroup of kernel of d_i and so we can form the quotient $H_i = \frac{\ker d_i}{\operatorname{im} d_{i+1}}$ called the ith **homology group** of the complex.

Note. The kernel of d_i is often called the i-**cycles** of the complex and denoted Z_i. The image of d_{i+1} is often called the i-**boundaries** of the complex and denoted B_i.

Definition (Chain map)**.** *Given two chain complices* (A, d) *and* (B, e) *we can form a map between them:* $f\colon A \to B$*. This is a set of maps* $f_i\colon A_i \to B_i$ *for each* i *such that the whole diagram commutes; that is for each* i *the following square commutes.*

$$\begin{array}{ccc} A_i & \xrightarrow{d_i} & A_{i-1} \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f_{i-1}} \\ B_i & \xrightarrow{e_i} & B_{i-1} \end{array}$$

Given a chain map, $f\colon C \to D$ there is an induced map between the homology groups of the chain complices.

Definition (Exact)**.** *If a sequence of abelian groups* $X \overset{f}{\to} Y \overset{g}{\to} Z$ *has the property* $\ker g = \operatorname{im} f$ *then we say the sequence is* ***exact*** *at* Y*. If a sequence*

$$X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_0$$

is exact for each i *then we call the sequence exact.*

The 5-term exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is called a ***short exact sequence****.*

Example.

- If the sequence $A \to B \to 0$ is exact at B then the map $A \to B$ is a surjection.
- If the sequence $0 \to A \to B$ is exact at A then the map $A \to B$ is an injection.
- If the sequence $0 \to A \to B \to 0$ is exact at A and B then $A \cong B$.
- If the sequence $0 \to A \to 0$ is exact at A then $A = 0$.

Given any map $f\colon A \to B$ there is a unique group K that is minimal with respect to making the sequence $K \to A \to B$ exact at A, clearly this is $\ker f$. That is, any other such group and morphism factors uniquely through K.

Similarly there is a unique minimal group making the sequence $A \to B \to C$ exact at B and this is called the **cokernel** of the f.

$$\operatorname{coker} f = \frac{B}{\operatorname{im} f}$$

2 Diagram chasing

Lemma 1 (3 × 3 lemma)**.** *Given a commutative diagram*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A'' & \longrightarrow & B'' & \longrightarrow & C'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with exact columns and exact bottom two rows, then the top row is also exact.

Proof. To show exact at A', need to show that $A' \xrightarrow{f} B'$ is injective.

$$\begin{array}{ccc}
A' & \xrightarrow[a \mapsto 0]{f} & B' \\
\Big\downarrow {\scriptstyle a \mapsto 0} & & {\scriptstyle 0 \mapsto 0} \Big\downarrow \\
A & \xrightarrow{0 \mapsto 0} & B
\end{array}$$

Since all the displayed maps in the diagram above are injective this shows $a = 0$.

To show exact at C', need to show that $B' \xrightarrow{g} C'$ is surjective.

Start with some $c' \in C$, this maps to some $c \in C$.

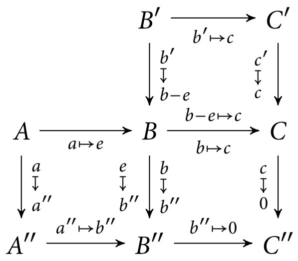
Now $B \to C$ is surjective so there is a $b \in B$ with $b \mapsto c \in C$.

Let b'' be the image of b in B''. Since c is in the image of $C' \to C$, we know $c \mapsto 0 \in C''$ and hence $b'' \mapsto 0 \in C''$.

Therefore b'' is in the image of $A'' \to B''$, so let $a'' \mapsto b''$ and take $a \in A$ in the preimage of a''.

Let $a \mapsto e \in B$, then $e \mapsto 0 \in C$ and hence $(b - e) \mapsto c$. Also $(b - e) \mapsto 0 \in B''$ and hence there is a $b' \in B$ such that $b' \mapsto (b - e)$.

Now since $C' \to C$ is injective we know that $b' \mapsto c$ and $B' \to C'$ is surjective.



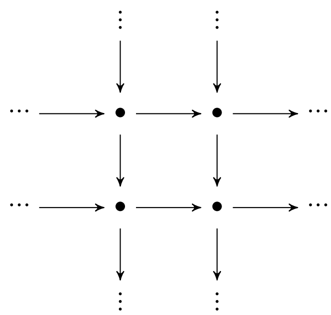
Exactness at B' is equally horrendous. ☐

3 A way out?

Much of this section is taken from [Ber12].

[Ber12] Bergman, *On diagram-chasing in double complexes*, Theory Appl. Categ. **26** (2012), No. 3, 60–96

Definition (Double complex)**.** *A* ***double complex*** *is a commutative diagram*

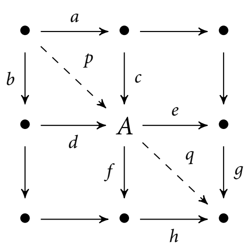


where each • is an abelian group; the composition of a consecutive pair of horizontal arrows is zero; and the composition of any consecutive pair of vertical arrows is zero.

That is, each row is a complex, each column is a complex and squares commute.

Example. The diagram from lemma 1 with zeros appended in all directions is an example of a double complex.

Definition (Homology, receptor and donor)**.** *Given a portion of a complex*



We define the familiar horizontal and vertical homology groups

$$A_{-\bullet-} = \frac{\ker e}{\operatorname{im} d} \qquad \textit{the } \textbf{\textit{horizontal homology}} \textit{ group}$$

$$A_{|\bullet|} = \frac{\ker f}{\operatorname{im} c} \qquad \textit{the } \textbf{\textit{vertical homology}} \textit{ group}$$

$${}^{\square}A = \frac{\ker e \cap \ker f}{\operatorname{im} p} \qquad \textit{the } \textbf{\textit{receptor}} \textit{ of } A$$

$$A_{\square} = \frac{\ker q}{\operatorname{im} c + \operatorname{im} d} \qquad \textit{the } \textbf{\textit{donor}} \textit{ of } A$$

Lemma 2 (Intramural maps)**.** *The identity map* id: $A \to A$ *induces maps*

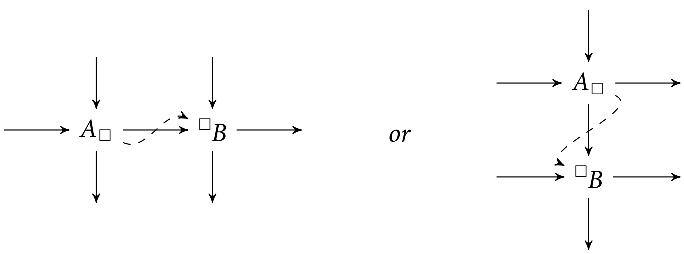
$$\begin{array}{ccc} {}^{\square}A & \longrightarrow & A_{-\bullet-} \\ \downarrow & & \downarrow \\ A_{|\bullet|} & \longrightarrow & A_{\square} \end{array}$$

Proof. For example:

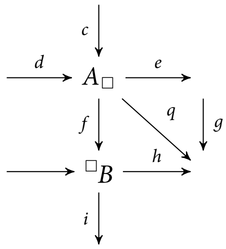
$${}^{\square}A = \frac{\ker e \cap \ker f}{\operatorname{im} p} \longrightarrow \frac{\ker e}{\operatorname{im} p} = \frac{\ker e}{\operatorname{im} db} \longrightarrow \frac{\ker e}{\operatorname{im} d} = A_{-\bullet-}$$

□

Lemma 3 (Extramural maps)**.** *An arrow in a double complex* $f\colon A \to B$ *induces a map* $A_{\square} \to {}^{\square}B$*, and hence the names donor and receptor.*



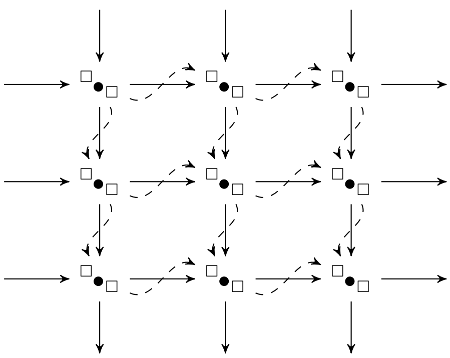
Proof. We prove the vertical case:



$$A_{\square} = \frac{\ker q}{\operatorname{im} c + \operatorname{im} d} \longrightarrow \frac{f(\ker q)}{f(\operatorname{im} c + \operatorname{im} d)} = \frac{f(\ker q)}{\operatorname{im} fd} \longrightarrow \frac{\ker h}{\operatorname{im} fd}$$
$$\longrightarrow \frac{\ker h + \ker i}{\operatorname{im} fd} = {}^{\square}B$$

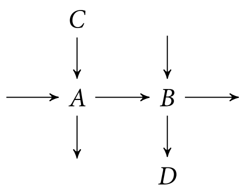
□

More globally, these maps look like



Notice that these maps cannot in general be composed as they appear head-to-head or tail-to-tail.

Lemma 4 (Salamander lemma)**.** *A portion of a double complex*



there is a six-term exact sequence

$$C_\square \dashrightarrow {}^\square A \dashrightarrow A \quad\quad C_\square \longrightarrow A\text{-}\bullet \longrightarrow A_\square \longrightarrow {}^\square B \longrightarrow B\text{-}\bullet \longrightarrow {}^\square D \quad\quad B\text{-}\bullet \dashrightarrow B_\square \dashrightarrow {}^\square D$$

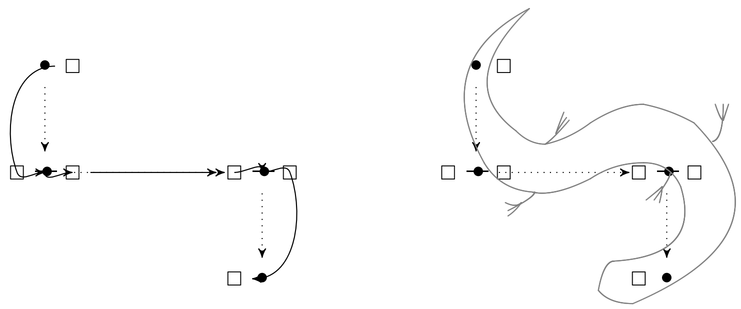
Similarly for a section of a double complex

$$\begin{array}{ccccc} & & \downarrow & & \\ C & \longrightarrow & A & \longrightarrow & \\ & & \downarrow & & \\ & \longrightarrow & B & \longrightarrow & D \\ & & \downarrow & & \end{array}$$

there is a six-term exact sequence

$$C_\square \dashrightarrow {}^\square A \dashrightarrow A \quad\quad C_\square \longrightarrow A{\scriptstyle\bullet}\!\!\mid \longrightarrow A_\square \longrightarrow {}^\square B \longrightarrow B{\scriptstyle\bullet}\!\!\mid \longrightarrow {}^\square D \quad\quad B{\scriptstyle\bullet}\!\!\mid \dashrightarrow B_\square \dashrightarrow {}^\square D$$

As a diagram this can be displayed as



which is supposed to look like a salamander (this diagram has been gratuitously stolen from [Ber12]).

[Ber12] Bergman, *On diagram-chasing in double complexes*, Theory Appl. Categ. **26** (2012), No. 3, 60–96

3.1 Important corollaries

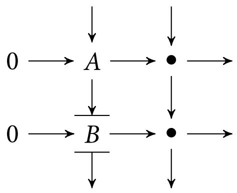
Though this lemma is a little technical there are two important corollaries that will be useful in diagram chasing.

Corollary 1. *For a horizontal arrow $A \longrightarrow B$, if the double complex is exact horizontally at both A and B then the induced map $A_\square \xrightarrow{\sim} {}^\square B$ is an isomorphism.*

Similary for a vertical arrow, and vertical exactness the induced morphism is an isomorphism.

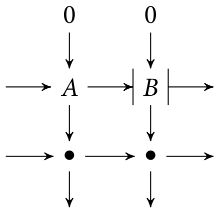
Corollary 2. *Given the displayed portion of a double complex exact in the way shown, then we have the isomorphisms to the diagram's right.*

For example the first diagram is exact horizontally at B.



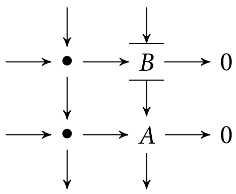
then

$${}^{\square}A \cong A\,{-}\!\bullet\!{-} \qquad A_{\square} \cong A\,{\mid}\!\!\bullet$$



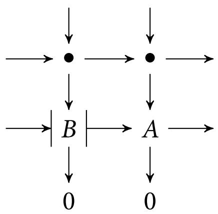
then

$${}^{\square}A \cong A\,{\mid}\!\!\bullet \qquad A_{\square} \cong A\,{-}\!\bullet\!{-}$$



then

$${}^{\square}A \cong A\,{\mid}\!\!\bullet \qquad A_{\square} \cong A\,{-}\!\bullet\!{-}$$



then

$${}^{\square}A \cong A\,{-}\!\bullet\!{-} \qquad A_{\square} \cong A\,{\mid}\!\!\bullet$$

Proof. We only prove one part as this gives the idea for the rest of the proof.

Take the first diagram and consider the salamander sequence for the arrow $0 \to A$. This gives

$$\bullet_{\square} \to 0 \to 0 \to {}^{\square}A \to A\,{-}\!\bullet\!{-} \to {}^{\square}B,$$

and ${}^{\square}B = 0$ by corollary 1; hence ${}^{\square}A \cong A\,{-}\!\bullet\!{-}$.

Similarly if we consider the salamander sequence for the arrow $A \to B$ we have

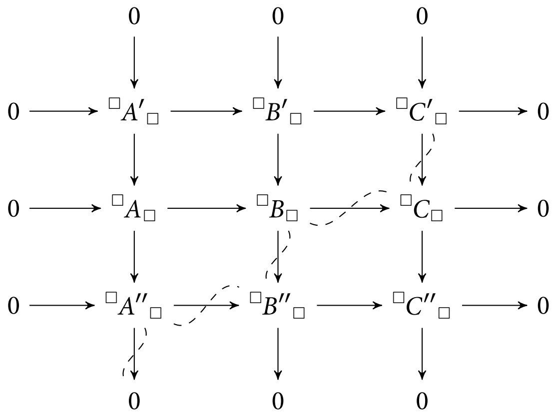
$$0 \to A\,{\mid}\!\!\bullet \to A_{\square} \to {}^{\square}B = 0 \to \cdots$$

and we have the second isomorphism. □

A simpler proof of the 3×3 lemma?

Proof. The first two diagrams of corollary 2 both apply to A' and hence ${}^{\square}A' \cong A'\,{-}\!\bullet\!{-} \cong A'_{\square} \cong A'\,{\mid}\!\!\bullet$, and we have the diagram is horizontally exact at A'.

Consider C': the second diagram in corollary 2 gives us $C' {-\!\bullet\!-} \cong C'_{\square}$, we can then repeatedly apply corollary 1 as shown.



$\square$